PLANETARY FLOW PATTERNS IN THE ATMOSPHERE

By C.-G. Rossby

The general purpose of this paper is to discuss the factors which determine the character of the prevailing flow patterns in the atmosphere, and in particular to bring out the conditions under which these flow patterns tend to remain stationary or to change. It has been written with the primary objective of presenting certain theoretical tools which appear to be helpful in the interpretation of and forecasting from consecutive weekly mean sea level pressure charts for the Northern Hemisphere. Maps of this type have been prepared at the Massachusetts Institute of Technology during the last few years in connection with a general research program sponsored by the U.S. Weather Bureau and aiming at the development of rational methods for the preparation of weather forecasts of a week's range or more.

The present article may be considered as an outgrowth of a speech before the joint meeting of the Royal Meteorological Society and the American Meteorological Society in Toronto in August, 1939. Some of the results presented at that time have already been published in the Journal of Marine Research (Rossby 1939) and will be discussed only very briefly below. Other ideas, barely touched upon at the Toronto meeting, have been developed further and will be treated in detail at the present time.

Most of the results presented below are readily obtained with the aid of Bjerknes' circulation theorem, which has been available to meteorologists for the last forty years. Under these circumstances it is rather startling that no systematic attempts have been made to study the planetary flow patterns in the atmosphere, particularly in view of the fact that far-reaching investigations of oceanic flow patterns were undertaken by Ekman a number of years ago (Ekman 1923).

The process of averaging pressures over periods of one week to some extent eliminates the horizontal pressure variations associated with such typically barocline phenomena as moving wave cyclones. The pressure systems of the weekly mean charts are of larger dimensions and correspond to the so-called "centers of action" in the atmosphere.

On the winter maps there are normally at least five such centers to be seen, the Icelandic and the Aleutian Lows, the Azores and the Asiatic Highs and finally the Pacific High, but one or several of these centers frequently breaks up into two parts. It is well known that these perturbations at higher levels no longer appear as closed isobaric systems but merely as undulations in the prevailing zonal pressure distribution.

In the following discussion these centers of action will be considered as quasi-permanent perturbations on the basic zonal (west-east) winds of the atmosphere. While the maintenance of these zonal winds obviously cannot be explained merely through an analysis of motions in a barotropic atmosphere, it will be shown below that a first indication of the factors that control the size and movements of the atmospheric centers of action may be obtained from a theoretical study of certain simple stationary and moving barotropic perturbations superimposed on a constant or slowly changing zonal wind system.

The ordinary gradient wind equation furnishes no clue to the interpretation of the horizontal pressure distributions (flow patterns) observed in the atmosphere. In particular, it leaves unanswered two basic questions, viz:

1. Do certain preferred flow patterns exist which are more readily established than others?

2. When will an arbitrarily prescribed flow pattern tend to remain stationary and when will it change or move?

The gradient wind equation is unable to help us answer these questions since it is *always* possible to find a theoretical wind blowing along the isobars and of such a speed that the corresponding deflecting and centrifugal forces precisely balance the horizontal pressure gradient. If, however, the resulting gradient wind field is studied with the aid of the equation of continuity, regions of horizontal convergence or divergence (pressure rise or pressure drop) will appear, indicating movements of or intensity changes in the initially observed pressure system. This combination of gradient wind equation and equation of continuity (J. Bjerknes 1937) has been of great value in stimulating research dealing with the nature of the horizontal pressure fields in the atmosphere and offers at least a partial answer to the second of the two questions posed above. The method does, however, suffer from some weaknesses. The most obvious is the assumption that gradient wind equilibrium prevails even though the systems studied generally are non-stationary. It is well known that in case of rapidly moving systems the neglected acceleration terms may reach the same order of magnitude as the centrifugal and deflecting forces.

The second and more basic objection to the procedure lies in the fact that it erroneously implies that the displacements of the observed flow patterns are *caused* by the isallobaric systems resulting from the horizontal convergences and divergences associated with the initial gradient wind distribution.

In the paper already referred to I have given an example of a wave motion without lateral boundaries which is purely horizontal and hence free from horizontal convergence or divergence, but which nevertheless may be non-stationary. The effect of lateral limits was studied by Haurwitz (1940). This example will be discussed further in this article to bring out the fact that the *factors determining the stationary or progressive character of the motion are to be found in the vorticity distribution* and that *the displacement of the pressure field is a secondary effect.*

In a barotropic atmosphere it is possible to eliminate the pressure between the two equations governing the horizontal components of motion. The resulting equation expresses the fact that vertical atmospheric columns, moving across the surface of the earth, must preserve their individual absolute vorticity after allowance has been made for such vorticity changes as may result from horizontal shrinking or stretching.

The absolute vorticity is made up of a vorticity relative to the rotating earth and of the vorticity of the earth's own rotation around the vertical. Since the latter factor varies with latitude it follows that the relative vorticity of a moving column must vary in a definite fashion with latitude. This variation in turn imposes certain restrictions on the radius of curvature of the trajectory described by the column. It follows that under steady or nearly steady conditions certain preferred flow patterns are established, merely as the result of the prescribed variation with latitude of relative vorticity.

To establish these patterns we shall consider an atmosphere consisting of several homogeneous, incompressible strata, each moving without change in its density. Thus we take the effect of stratification into account, at least in a crude fashion, while leaving out of consideration the effects of compressibility. Within each of these homogeneous layers, the equations for frictionless motion are

$$(1) \qquad \frac{du}{dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$(2) \qquad \frac{dv}{dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

and the equation of continuity

$$(3) \qquad \frac{dD}{dt} = -D\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right),$$

u and v being the horizontal velocity components, $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$ the components of the horizontal pressure gradient and D the depth of a vertical fluid column extending throughout the entire homogeneous layer under discussion. It is evident that this formulation of the equation of continuity implies that the horizontal velocity components do not vary with elevation within the layer D. The symbol f represents the Coriolis parameter, $2\Omega \sin \phi$, where ϕ is the latitude and Ω the angular velocity of the earth.

It will be assumed that the y-axis points northward, the x-axis eastward. Thus

$$(4) \qquad \frac{df}{dt} = v\frac{\partial f}{\partial y} = \beta v,$$

in which equation β is defined by

$$(5) \qquad \beta = \frac{\partial f}{\partial y} = \frac{2\Omega \cos \phi}{a},$$

a being the radius of the earth. The quantity β has been computed previously (Rossby 1939) and will be treated as a constant below. The table for β is reproduced for the convenience of the reader (Table I).

TABLE I

VARIATION OF β WITH LATITUDE

Latitude	$\beta . 10^{13} cm^{-1} sec^{-1}$
90°	0.0
75°	0.593
60°	1.145
45°	1.619
30°	1.983
15°	2.212
0°	2.290

If the relative vorticity is designated by ζ, so that

$$(6) \qquad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

one finds through cross-differentiation of (1) and (2) and after more simplifications,

$$\frac{d\zeta}{dt} = -(f+\zeta)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) - v\beta \tag{7}$$

$$\frac{d(f+\zeta)}{dt} = -(f+\zeta)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \tag{8}$$

and finally, with the aid of the equation of continuity

$$\frac{d}{dt}\left(\frac{f+\zeta}{D}\right) = 0. \tag{9}$$

This analysis is based on the assumption that the terms $w\frac{\partial u}{\partial z}$ and $w\frac{\partial v}{\partial z}$ may be neglected within each layer, an assumption we know to be correct in the steady state, since the gradient wind is independent of elevation in a barotropic medium, but the assumption is probably well justified also in large-scale non-stationary barotropic systems. Thus the absolute vorticity, $f+\zeta$, must be proportional to the depth of the fluid column as it moves over the surface of the earth,

$$f+\zeta = c.D, \tag{10}$$

where c is a constant which may vary from one trajectory to the next. The last integral may be written

$$\zeta = \zeta_0 + (f_0 - f) + (f_0 + \zeta_0)\,.\,\frac{D-D_0}{D_0}, \tag{11}$$

which connects initial conditions (subscript 0) and later conditions (no subscript) of the same fluid column. In the absence of depth variations this equation reduces to

$$\zeta + f = \zeta_0 + f_0, \tag{12}$$

an equation which was used previously to explain the behaviour of perturbations on the zonal west winds of the atmosphere.

If the atmosphere consists of a finite number of layers of constant potential temperature, each having a finite thickness and moving adiabatically and all of them arranged in stable position so that the potential temperature increases from one layer to the one next above, then the equations of motion change into

$$\frac{du}{dt} = fv - \frac{\partial \pi}{\partial x} \tag{1b}$$

$$\frac{dv}{dt} = -fu - \frac{\partial \pi}{\partial y} \tag{2b}$$

($\pi = c_p T$, c_p = specific heat)
(T = absolute temperature)

and the equation of continuity into

$$\frac{d\Delta}{dt} = -\Delta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right), \tag{3b}$$

where Δ now measures the weight per unit cross-section of an individual vertical air column of the layer in question, expressed as the difference in pressure between the bottom and the top of the column. In the same fashion as before one obtains

$$\frac{d}{dt}\left(\frac{f+\zeta}{\Delta}\right) = 0. \tag{9b}$$

It is possible to derive corresponding results also for an atmosphere in which the potential temperature varies continuously with elevation and in which all displacements are adiabatic but the preceding demonstration is adequate for our purposes. The generalized treatment will be presented in another place.

Integration of (9b) gives

$$f+\zeta=c.\Delta \tag{10b}$$

or

$$\zeta=\zeta_0+(f_0-f)+(f_0+\zeta_0)\,\frac{\Delta-\Delta_0}{\Delta_0}\,. \tag{11b}$$

In the application of (11) and (11b) to a medium with continuous density or potential temperature variation along the vertical it becomes necessary to consider infinitesimal sheets of air limited by adjacent isopycnal surfaces ($\rho=$constant, $\rho+\delta\rho=$constant) or by adjacent isentropic surfaces ($\theta=$constant, $\theta+\delta\theta=$constant). The assumption is then made that the fluid inside such a sheet is characterized by a constant mean density, $\bar{\rho}=\rho+\frac{1}{2}\delta\rho=$constant, or by a constant mean potential temperature, $\bar{\theta}=\theta+\frac{1}{2}\delta\theta=$constant. *In the computation of the relative vorticity* ζ *it must then be remembered that the operators* $\frac{\partial}{\partial x}$ *and* $\frac{\partial}{\partial y}$ *refer to variations with respect to x or y inside a layer of constant density or constant potential temperature and really should be written*

$$\left(\frac{\partial}{\partial x}\right)_\rho,\ \left(\frac{\partial}{\partial y}\right)_\rho \text{ or } \left(\frac{\partial}{\partial x}\right)_\theta,\ \left(\frac{\partial}{\partial y}\right)_\theta$$

in accordance with the procedure adopted in thermodynamics. A rigid proof of this statement will be furnished in another place.

The two equations (11) and (11b) state that the relative vorticity of a fluid column at any time is equal to the sum of 1) the initial vorticity of the column, 2) a term (f_0-f) representing gain in relative vorticity due to displacements along the meridians and 3) a term representing gain due to vertical stretching.

The constant occurring in (10) and (10b) is obviously a characteristic of the particular air column to which the analysis pertains. It has been suggested by my collaborator, Mr. V. P. Starr, that this characteristic constant which presumably varies from one fluid element to another might be used for identification purposes in the analysis of air masses. A report on this ingenious method of utilizing the vorticity theorem will be given by Messrs. Starr and Neiburger (Starr and Neiburger, 1940) in another place. Here it is enough to mention that the constant c, the physical meaning of which is not very clear, may be replaced by the ζ_0 in (11) or (11b). *This quantity, which may be called the potential vorticity, represents the vorticity the air column would have if it were brought, isopycnically or isentropically, to a standard latitude* (f_0) *and stretched or shrunk vertically to a standard depth* D_0 *or weight* Δ_0.

It is convenient to choose for f_0 a value of exactly 10^{-4} sec.$^{-1}$, corresponding to a latitude of about 43°N. If one considers a sheet of air limited by isentropic surfaces 4° or 5° apart it is convenient to choose Δ_0 equal to 100 mb.

In order to interpret the results of the preceding analysis, it is necessary to express the relative vorticity in a physically readily understood form. It is well known that in case of cyclonically curved *streamlines*

$$\zeta=\frac{c}{R}+\frac{\partial c}{\partial r}, \tag{13}$$

where c is the wind velocity, R is the radius of curvature of the streamline at the point (P) for which the vorticity is to be determined and $\frac{\partial c}{\partial r}$ is the rate of shear of the wind at the point (P), the coordinate r being counted from the centre of curvature toward P.

In the case of anticyclonically curved streamlines one finds

$$\zeta = -\left(\frac{c}{R} + \frac{\partial c}{\partial r}\right). \tag{14}$$

However, expression (13) may be used for both cyclonically and anti-cyclonically curved streamlines, provided both R and r always are measured from the centre of curvature and counted positive from left to right when facing downstream.

The two expressions (13) and (14) show that vorticity may express itself either as shear or as a curvature of the streamline. This point has been discussed in detail by Ekman (1932).

We shall next apply the vorticity theorem derived above to a study of the dynamic stability of deep westerly and easterly winds and of the nature of the flow patterns that result from the application of small disturbing impulses to such current systems. For this purpose we shall consider a steady, narrow air current in an isentropic atmosphere, surrounded on both sides by resting air. In view of the barotropic nature of the system, the winds do not vary with elevation.

Next we select an arbitrary horizontal curve S, consisting of sections across the narrow current at two points, 1 and 2, with connecting portions of the curve in the surrounding resting fluid. Section 2 is downstream from Section 1. We know that

$$\frac{d}{dt}\left(\frac{f+\zeta}{\Delta}\right) = 0 \tag{9b}$$

for all the fluid columns enclosed by S and thus

$$\iint_A \Delta \frac{d}{dt}\left(\frac{f+\zeta}{\Delta}\right) . \ \delta x \delta y = 0 \tag{15}$$

Here the integration extends over the entire area A enclosed by S.

It follows from the assumption of steady motion that (15) may be transformed into

$$\iint_A \left[u\Delta \frac{\partial}{\partial x}\left(\frac{f+\zeta}{\Delta}\right) + v\Delta \frac{\partial}{\partial y}\left(\frac{f+\zeta}{\Delta}\right)\right] \delta x \delta y = 0 \tag{16}$$

or, after integration by parts,

$$\iint_A \left[\frac{\partial}{\partial x}\left\{u(f+\zeta)\right\} + \frac{\partial}{\partial y}\left\{v(f+\zeta)\right\}\right] \delta x \delta y - \tag{17}$$

$$-\iint_A \frac{f+\zeta}{\Delta}\left(\frac{\partial u\Delta}{\partial x} + \frac{\partial v\Delta}{\partial y}\right) \delta x \delta y = 0.$$

The second integral vanishes because of the requirement of continuity and the assumption of a steady state. Transforming the first surface integral into a line integral, one obtains

$$\int_A c_n(f+\zeta)\,.\,\delta\sigma=0, \tag{18}$$

$\delta\sigma$ being a line element of S and c_n the outward velocity component norma to S. We know that c_n vanishes in the portions of S which fall within the resting fluid and thus it follows that

$$\int_{(1)} c(f+\zeta)\delta\sigma=\int_{(2)} c(f+\zeta)\delta\sigma=\text{constant}, \tag{19}$$

or, the *absolute vorticity transport* across any section of the current is constant. In this expression c is the downstream velocity.

It follows from a combination of (13) and (19) that

$$\int c\left(f+\frac{c}{R}+\frac{\partial c}{\partial r}\right)\delta r=\text{constant}, \tag{20}$$

For sufficiently narrow current systems it is permissible to assume R to be constant across the current and thus it follows that

$$f\int_{r_1}^{r_2} c\,\delta r+\frac{1}{R}\int_{r_1}^{r_2} c^2\delta r=\text{constant}, \tag{21}$$

the term contributed by the shear $\frac{\partial c}{\partial r}$ vanishing since $c=0$ for $r=r_1$ and for $r=r_2$. These two values represent the boundaries of the current in any one section. It is evident that

$$\int_{r_1}^{r_2} c\,\delta r=V \tag{22}$$

represents the volume transport of the current per unit depth and

$$\int_{r_1}^{r_2} c^2\,\delta r=M \tag{23}$$

the momentum transport per unit depth and for unit density. One finds then, that

$$\boxed{\text{Vorticity transport}=fV+\frac{M}{R}=\text{constant}} \tag{24}$$

along the trajectory of the current. For anticyclonically curved systems

$$\boxed{\text{Vorticity transport}=fV-\frac{M}{R}=\text{constant.}} \tag{25}$$

Both equations express the fact that the total absolute vorticity transport remains constant from section to section along the current. Equation (24) can be used for both cyclonic and anticyclonic trajectories provided one adopts the convention that R is positive in the former, negative in the latter case.

The two equations (24) and (25), expressing the constancy of the vorticity transport, apply also to the vorticity transport between two adjacent stream lines, provided the shear can be neglected. Thus these equations should be applicable to the motion near the axes of the broad atmospheric current systems.

The constant on the right side in (24) or (25) may be given another interpretation. In case of a steady, curved and narrow stream (*cyclonic curvature*) in an isentropic atmosphere, the balance of forces normal to the current axis is expressed through the equation

$$fc+\frac{c^2}{R}=\frac{1}{\rho_0}\frac{\partial p_0}{\partial r}, \tag{26}$$

p_0 being sea level pressure, ρ_0 sea level density, and the transversal coordinate r being counted from the centre of curvature toward the streamline. The sea level density is constant along a streamline (isobar). If cross-current variations in R, the radius of curvature of the streamlines, are neglected, it follows after integration across the current that

$$\text{Vorticity transport}=fV+\frac{M}{R}=\frac{\Delta p_0}{\bar{\rho}_0} \qquad \text{(cyclonic case)} \tag{27}$$

Δp_0 being the total (positive) pressure difference across the current and $\bar{\rho}_0$ the mean sea level density across the current.

It follows from a comparison of (24) and (27) that this pressure difference must remain constant along the trajectory of the steady current. The reasoning is easily extended to the anticyclonic case which gives

$$\text{Vorticity transport}=fV-\frac{M}{R}=\frac{\Delta p_0}{\bar{\rho}_0} \qquad \text{(anticyclonic case)} \tag{28}$$

provided R is counted positive also in this case.

The absolute vorticity transport of an anticyclonically curved current vanishes when Δp_0 vanishes and our fundamental equation then reduces to the form

$$R=\frac{M}{fV}=\frac{\bar{c}}{f}, \qquad \bar{c}=\frac{M}{V}=\frac{\int_{r_1}^{r_2} c^2\delta r}{\int_{r_1}^{r_2} c\,\delta r}, \tag{29}$$

where $\bar{c}$ is a mean velocity. This equation is identical with the equation for the inertia path, which thus appears as one of the permissible trajectories of constant vorticity transport, satisfying the particular requirement of zero absolute vorticity transport.

We shall make use of (27) and (28) to investigate the stability of deep westerly and easterly winds, respectively. The vorticity transport of a narrow, straight west wind current is given by the expression f_0V, f_0 being the value of the Coriolis' parameter at the latitude of the current axis ($y=0$). If at a point $x=0$ the moving current is deflected northward, the streamline must possess cyclonic curvature ($R>0$) in a narrow region $x>0$. The vorticity transport in this same region has the value

$$fV+\frac{M}{R}=(f_0+\beta y)V+\frac{M}{R}>f_0V. \tag{30}$$

It is easily seen that the extent to which the current can push northward is limited by the cyclonic vorticity of the initial perturbation. As y increases $\frac{1}{R}$ must decrease and finally vanish (inflexion point). For still larger y-values $\frac{1}{R}$ becomes negative and the current acquires an anti-cyclonic curvature which brings it back toward the initial position. *Thus, in this sense, such a narrow, deep west wind current is stable.* The initial cyclonic vorticity of the perturbation is soon checked by the increasing anticyclonic vorticity associated with the northward displacement; downstream from the initial perturbation point the current will simply tend to oscillate around an equilibrium position. The differential equation for this stable perturbation pattern is best obtained by shifting the origin ($y=0$) to the mean position of the disturbed current. This mean position must obviously coincide with the inflexion points in the current. Thus

$$fV+\frac{M}{R}=f_0V \tag{31}$$

is the equation for the current, provided R is positive for cyclonic curvature, negative for anticyclonic. From this equation it follows that

$$\beta yV+\frac{M}{R}=0 \tag{32}$$

and, since

$$\frac{1}{R}=\frac{d^2y}{dx^2}\cdot\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-\frac{3}{2}}, \tag{33}$$

$$\beta yV+M.\frac{d^2y}{dx^2}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-\frac{3}{2}}=0. \tag{34}$$

A first integration of (34) is easily accomplished. Through the substitution

$$\frac{d^2y}{dx^2}=q.\frac{dq}{dy},\ \left(q=\frac{dy}{dx}\right) \tag{35}$$

one obtains

$$\frac{\beta V}{M}ydy+\left(1+q^2\right)^{-\frac{3}{2}}.\,qdq=0 \tag{36}$$

or

$$\frac{\beta V}{M}y^2-\frac{2}{\sqrt{1+q^2}}=\text{constant}. \tag{37}$$

If y_0 signifies the northernmost displacement of the current from its mean position ($y=0$) one finds

$$\frac{\beta V}{M}(y_0^2-y^2)-2\left[1-\frac{1}{\sqrt{1+q^2}}\right]=0. \tag{38}$$

For long perturbations of *small* amplitude ($|q|<<1$) this gives

$$q^2=\frac{\beta V}{M}(y_0^2-y^2), \tag{39}$$

or

(40) $$\frac{ds}{dx}=\sqrt{\frac{\beta V}{M}}\sqrt{1-s^2}, \qquad s=\frac{y}{y_0}.$$

This equation is readily integrated and gives

(41) $$s=\frac{y}{y_0}=\sin\sqrt{\frac{\beta V}{M}}x=\sin x\sqrt{\frac{\beta}{\bar{c}}}.$$

Here x is counted from the point where s vanishes. It is obvious that (41) corresponds to a simple harmonic stationary wave. The wave length L is given by the relation

(42) $$\sqrt{\frac{\beta}{\bar{c}}}.L=\sqrt{\frac{\beta V}{M}}.L=2\pi$$

or

(43) $$L=2\pi\sqrt{\frac{M}{\beta V}}=2\pi\sqrt{\frac{\bar{c}}{\beta}},$$

where $\bar{c}$ is the mean velocity defined in (29). This expression for the wave length of a stationary perturbation on a narrow west wind current is in perfect agreement with the result obtained earlier (Rossby 1939) for a stationary perturbation on an infinitely broad west wind and appears thus to have a considerable broad general validity. For convenience, a table for this stationary wave length L as function of $\bar{c}$ and β (latitude) is reprinted below from the earlier report (upper figure in each box of Table II).

TABLE II

STATIONARY WAVELENGTH L (IN KM) AND DIMENSION h (IN KM) OF STATIONARY PLANETARY EDDIES AS FUNCTIONS OF ZONAL WIND VELOCITY $\bar{c}$ AND LATITUDE φ. TOP FIGURE GIVES L, LOWER FIGURE GIVES h.

φ \ $\bar{c}$	4 m/sec	8 m/sec	12 m/sec	16 m/sec	20 m/sec
0°	2626 591	3714 836	4548 1024	5252 1182	5872 1322
15°	2672 601	3779 850	4628 1042	5344 1203	5974 1345
30°	2822 635	3990 898	4888 1100	5644 1270	6310 1420
45°	3120 703	4412 994	5405 1218	6241 1406	6978 1572
60°	3713 836	5252 1182	6432 1448	7428 1672	8304 1869
75°	5160 1162	7298 1643	8938 2012	10321 2323	11539 2597

It is evident that the waves of small amplitude here discussed intersect the x-axis at a very small angle. Another extreme case is provided by a current intersecting the axis at right angles. It then follows that

$$\frac{dy}{dx}=q=\infty \text{ for } y=0$$

and hence, from (38),

$$\frac{\beta V}{M} y_0^2 = 2, \quad y_0 = 1.414 \sqrt{\frac{c}{\beta}}. \tag{44}$$

Substitution gives

$$\frac{y^2}{y_0^2} = \frac{1}{\sqrt{1+q^2}}. \tag{45}$$

A quarter of a wave length is obtained by integration between $y=0$ and $y=y_0$. The result is

$$L = 4y_0 \int_0^1 \frac{s^2 ds}{\sqrt{1-s^4}} = 5.656 \sqrt{\frac{c}{\beta}} \int_0^1 \frac{s^2 ds}{\sqrt{1-s^4}} = 3.39 \sqrt{\frac{c}{\beta}} \tag{46}$$

It is apparent that this stationary wave of finite amplitude is considerably shorter than the corresponding stationary wave of very small amplitude. In figure 1, both types of waves have been entered for comparison. The two solutions compared are

$$\frac{x}{\sqrt{\frac{c}{\beta}}} = \sqrt{2} \int_0^s \frac{s^2 ds}{\sqrt{1-s^4}} \text{ and } \frac{x}{\sqrt{\frac{c}{\beta}}} = \text{arc sin } s.$$

In the first of these curves

$$s = \frac{y}{y_0}, \quad y_0 = \sqrt{2} \cdot \sqrt{\frac{c}{\beta}}, \quad s\sqrt{2} = \frac{y}{\sqrt{\frac{c}{\beta}}}$$

in the second the value of y_0 is arbitrary but obviously small in comparison with $\sqrt{2}\sqrt{\frac{c}{\beta}}$. The non-dimensional coordinates used in figure 1 are

$$\frac{x}{\sqrt{\frac{c}{\beta}}} \text{ and } \frac{y}{\sqrt{\frac{c}{\beta}}}.$$

In the case of a narrow easterly current flowing through resting fluid along a fixed latitude circle the vorticity transport is constant and given by $f_0 V$. If the current is disturbed and deflected slightly southward, west of a certain longitude, it follows that the current will have a slight cyclonic curvature. Its vorticity transport now is given by $fV + \frac{M}{R}$, where f is smaller than f_0, the current all the time moving toward more southerly latitudes. As the current continues southward, f decreases, $\frac{1}{R}$ increases, and the current assumes a steadily decreasing radius of curvature away from its original direction. Finally the current has turned around completely and begins to move north, while still turning around cyclonically. Ultimately it reaches its original latitude and continues westward.

It appears from this analysis that a slight deflection southward would cause an initially straight, narrow current from the east to make a complete cyclonic circuit. It is likewise easy to see that a slight initial deflection northward would cause the current to describe a complete anticyclonic

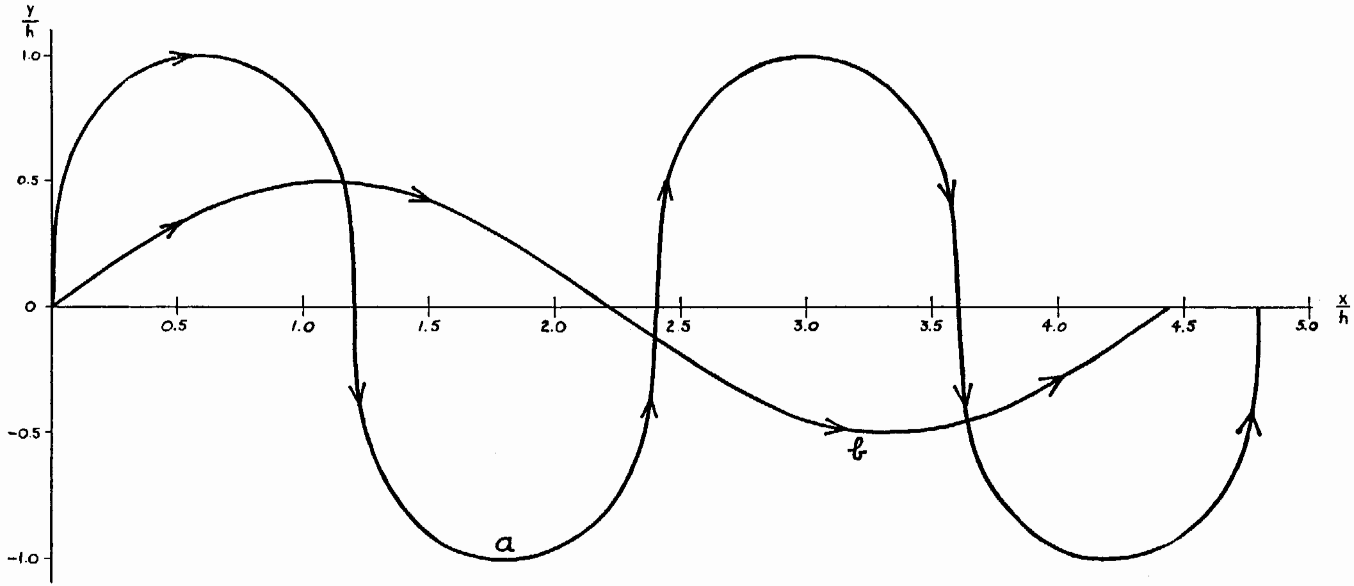

FIG. 1

Comparison of stationary waves in currents of finite, constant width, for large and for small amplitudes. Curve *a* represents a stationary flow pattern, the current intersecting its mean latitude at right angles. Curve *b* represents a stationary wave pattern of small amplitude and has the nature of a pure sine curve. In the diagram the amplitude of this latter wave is greatly exaggerated. The ratio between the two stationary wave-lengths is about 0.55. Geometrically these flow patterns are similar to inertia paths at the equator but physically they are of a different nature and can appear in any latitude.

circuit. It would thus appear justifiable to state that *easterly wind currents in such a barotropic atmosphere are unstable and* must break up into intermittently re-established cyclonic or anticyclonic vortices as a result of very small impressed forces.

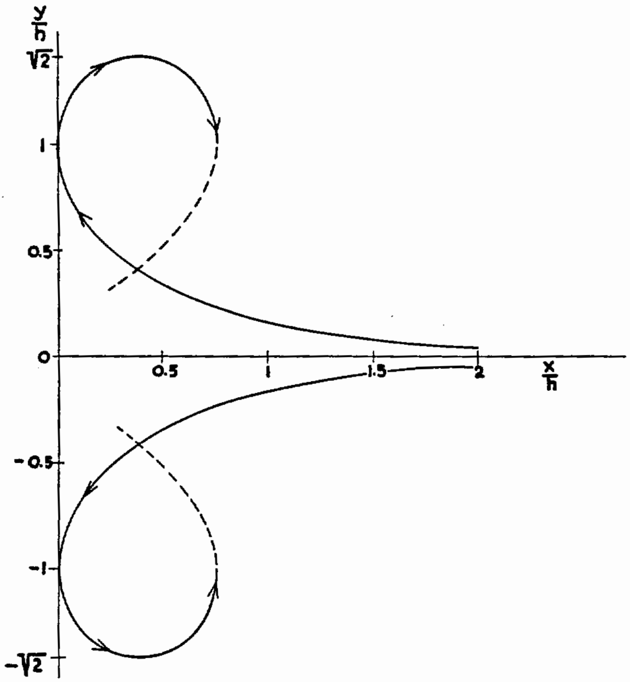

FIG. 2

Stationary cyclonic and anticyclonic eddy patterns of limited width resulting from the breaking up of dynamically unstable easterly winds. Geometrically these flow patterns are similar to inertia paths at the equator, but physically they are of a different nature and can appear in any latitude.

The cyclonic and anticyclonic trajectories described above are represented in figure 2. The mathematical analysis is simple. We place the x-axis along the latitude of the undisturbed current and count x positive eastward from the longitude where the rotating current appears as a pure south wind (anticyclonic case). The equation for the vorticity transport is

$$fV - \frac{M}{R} = f_0 V \tag{47}$$

or

$$\beta y V = \frac{M}{R}. \tag{48}$$

Inserting the value for the radius of curvature one obtains

$$\beta y V = M\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-\frac{3}{2}} \cdot \frac{d^2y}{dx^2}. \tag{49}$$

We shall introduce a length h defined by

$$h=\sqrt{\frac{2M}{\beta V}}=\sqrt{\frac{2\bar{c}}{\beta}}. \tag{50}$$

and also make use of the substitution (35). Then it follows that (49) changes into

$$\frac{2ydy}{h^2}=\left(1+q^2\right)^{-\frac{3}{2}}qdq \qquad \left(q=\frac{dy}{dx}\right) \tag{51}$$

or, if

$$\eta=\frac{y}{h},\ \xi=\frac{x}{h},\ q=\frac{dy}{dx}=\frac{d\eta}{d\xi}, \tag{52}$$

$$4\eta d\eta=\left(1+q^2\right)^{-\frac{3}{2}}dq^2. \tag{53}$$

Integration gives

$$2\eta^2=-\frac{2}{\sqrt{1+q^2}}+K,\ (K \text{ is a constant}). \tag{54}$$

For large x-values η and q must both vanish. Thus

$$0=-2+K \tag{55}$$

and

$$\frac{1}{\sqrt{1+q^2}}=1-\eta^2. \tag{56}$$

At the point where south wind prevails q is infinitely large. It follows from (56) that the current has turned 90° at a distance $\eta=1$ or $y=h$ to the north of its original latitude.

We may now solve for q and find

$$q=-\frac{\eta\sqrt{2-\eta^2}}{1-\eta^2}=\frac{d\eta}{d\xi}. \tag{57}$$

Integration gives

$$\xi=\int_\eta^1\frac{(1-\eta^2)\,d\eta}{y\sqrt{2-\eta^2}} \qquad (0\leq\eta\leq1) \tag{58}$$

The upper half of the trajectory is likewise obtained from (48) but the absolute value of the radius of curvature is now obtained from

$$\frac{1}{R}=-\frac{d^2y}{dx^2}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-\frac{3}{2}} \tag{59}$$

Thus, after integration

$$2\eta^2=\frac{2}{\sqrt{1+q^2}}+K'. \tag{60}$$

It has already been shown that

$$\eta = 1 \text{ for } q = \infty. \tag{61}$$

Thus,

$$\eta^2 = \frac{1}{\sqrt{1+q^2}} + 1 \tag{62}$$

and

$$q = \frac{\eta\sqrt{2-\eta^2}}{\eta^2 - 1} = \frac{d\eta}{d\xi} \tag{63}$$

Integration gives

$$\xi = \int_1^\eta \frac{\eta^2 - 1}{\eta\sqrt{2-\eta^2}} d\eta \qquad (1 \leq \eta \leq \sqrt{2}).$$

It follows from (63) that the current blows from the west at the point where q vanishes, that is, at a point where $\eta = \sqrt{2}$ or $y = h\sqrt{2}$. The dimension of this trajectory as measured by h is tabulated in table 2 (lower number in each box).

Up to this point we have been dealing with narrow currents flowing through a resting medium and this method of attack has enabled us to study stationary or, in the case of east winds, intermittently re-established quasi-stationary patterns of finite dimensions. The wave patterns studied in the first report were all of small amplitude, since it was assumed that second order terms could be neglected in the mathematical analysis. The fact that both theoretical attacks lead to the same value for the wave length of stationary waves should materially strengthen the confidence in the result. On the other hand, it is quite evident that a narrow current must be subject to rapid mixing with its environment and thus gradually lose its identity. Thus the principal value of the preceding analysis of narrow currents lies in the fact that it brings out the basic difference between the stability of easterly and westerly winds.

We shall next attempt to answer the second question presented in the introduction to this paper; namely, when will an arbitrarily prescribed flow pattern tend to remain stationary and when will it change or move? It is well known from the theory for absolute two-dimensional motion in ideal fluids that the vortex filaments, which in this case extend normal to the plane of motion, are carried along as conservative properties by the moving fluid. In our case the *potential vorticity* is conservative and carried along by the fluid.

Under stationary conditions streamlines and trajectories coincide, and along such a stationary trajectory potential vorticity must remain constant. Thus, an observed state of motion in a barotropic sheet in the atmosphere can be steady only if lines of constant potential vorticity and streamlines coincide. If they intersect, the relative orientation of the two sets of lines permits one to predict the instantaneous direction of change in the potential vorticity at any given point and, if depth changes can be neglected, also in the actual relative vorticity.

We shall illustrate this by an application to the simple two-dimensional wave analyzed in my previous report.

The velocity distribution is given by a positive constant zonal velocity U

along the x-axis (pointing eastward) and by a perturbation velocity v' along the y-axis (pointing northward). This velocity is given by

$$v' = v_0' \cos\frac{2\pi}{L}(x - ct). \tag{64}$$

The wave velocity c has the value

$$c = U - \frac{\beta L^2}{4\pi^2} \tag{65}$$

The streamlines may be computed from

$$\frac{dx}{U} = \frac{dy}{v'} \tag{66}$$

and one finds, for the time $t=0$

$$y_{\text{stream line}} = \frac{v'_0 L}{2\pi U}\sin\frac{2\pi x}{L}.$$

The trajectory of the particle which at the time $t=0$ passed through the point $x=y=0$ is given by integration of the simultaneous equations

$$\frac{dx}{dt} = U, \quad \frac{dy}{dt} = v'_0 \cos\frac{2\pi}{L}(x - ct). \tag{67}$$

It follows that

$$\frac{dy}{dt} = v'_0 \cos\frac{2\pi}{L}(U - c)t \tag{68}$$

and

$$y = \frac{v_0' L}{2\pi(U-c)}\sin\frac{2\pi}{L}(U-c)t \tag{69}$$

or

$$y_{\text{trajectory}} = \frac{v'_0 L}{2\pi(U-c)}\sin\frac{2\pi}{L}\frac{U-c}{U}x. \tag{70}$$

The potential vorticity ζ_0 is defined by

$$\zeta_0 = \zeta + f - f_0 = \zeta + \beta y = \frac{\partial v'}{\partial x} + \beta y. \tag{71}$$

It follows that

$$\zeta_0 = \beta y - \frac{2\pi v_0'}{L}\sin\frac{2\pi}{L}(x - ct). \tag{72}$$

The particular line on which the potential vorticity vanishes is given by

$$y_{\text{pot. verocity}} = \frac{2\pi v_0'}{\beta L}\sin\frac{2\pi}{L}(x - ct) \tag{73}$$

and, at time $t=0$ by

$$y_{\text{pot. verocity}} = \frac{2\pi v_0'}{\beta L}\sin\frac{2\pi x}{L} \tag{74}$$

The following points of interest are brought out by this analysis:

A. The ratio of the amplitude of the trajectory to the amplitude of the streamline is $\frac{U}{U-c}$. Thus individual particles move farther north and south than the streamlines seem to suggest when the waves move eastward, and move less far north and south than the streamlines seem to suggest in the case of retrograde waves.

B. At a prescribed time, streamlines and lines of constant potential vorticity coincide if

$$\frac{v_0'L}{2\pi U} = \frac{2\pi v_0'}{\beta L} \tag{75}$$

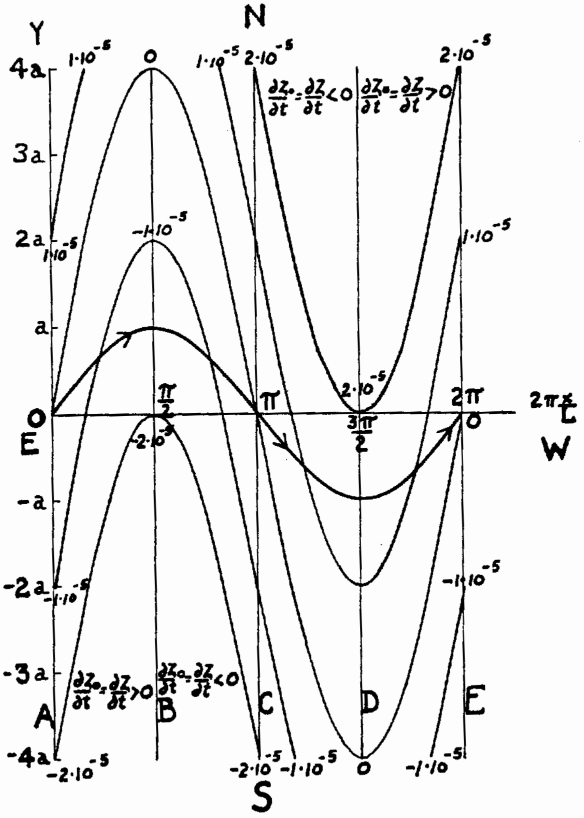

FIG. 3a

Relative orientation of stream lines and lines of constant potential vorticity in short wave ($L=\frac{1}{2}L_s$). The amplitude of the wave, a, has been so chosen that $\beta a = 0.5\ 10^{-5}\text{sec}^{-1}$, and consequently the potential vorticity increases northward by that amount in the distance a. It is seen that the two sets of lines intersect in such a fashion that the potential (and hence the actual) vorticity must decrease between *B* and *D*, increase on both sides thereof. It is easily seen that as a result, the line of zero vorticity (*C*) must be displaced eastward, i.e. the wave is progressive.

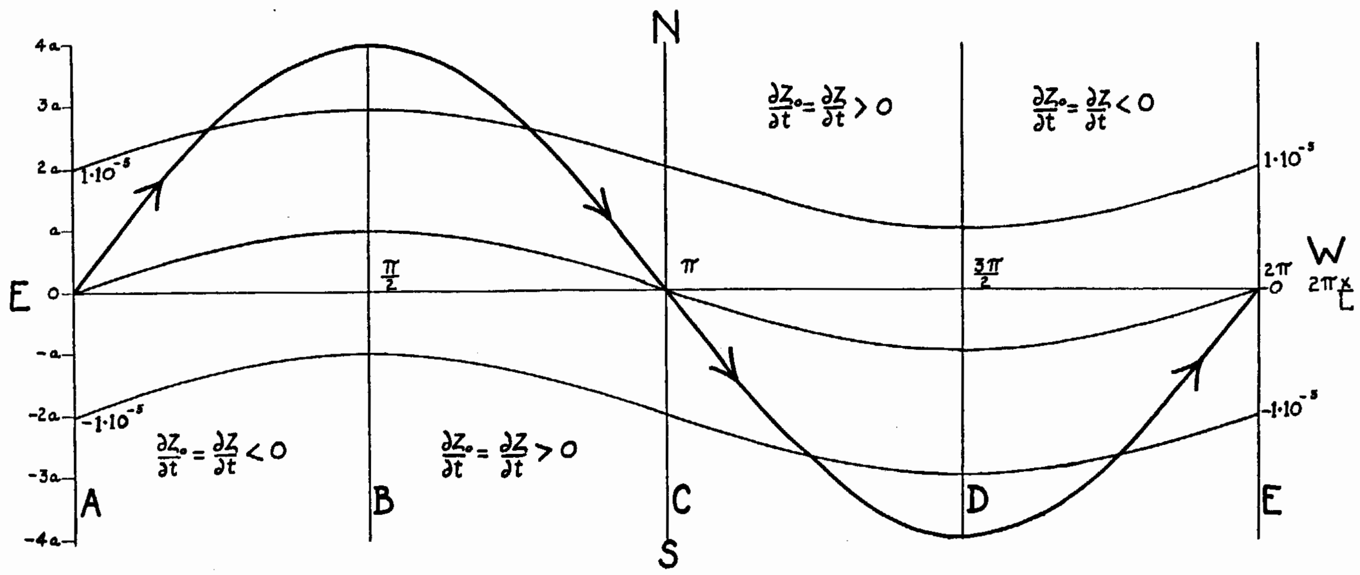

FIG. 3b

Relative orientation of stream lines and lines of constant potential vorticity in long wave ($L=2Ls$). In this case the amplitude was chosen four times as large as in the case of the short wave (Fig. 3a), and again the potential vorticity increases northward by the amount 0.5 10^{-5}sec^{-1} in the distance a. It is seen that the two sets of lines intersect in such a fashion that the potential (and hence actual) vorticity must increase between B and D, decrease on both sides thereof. As a result, the line of zero vorticity (C) must be displaced westward, i.e. the wave is retrograde.

or

$$L = L_s = 2\pi \sqrt{\frac{U}{\beta}}. \tag{76}$$

which is precisely the condition for stationary waves.

The ratio between the amplitudes of the constant potential vorticity lines and of the streamlines is

$$\frac{4\pi^2 U}{\beta L^2} = \frac{U}{U-c} \tag{77}$$

and thus the potential vorticity lines have greater amplitude than the streamlines for short waves ($L<L_s$) and vice versa for long waves ($L>L_s$). The three cases are illustrated in figure 3. The relative orientation of the two sets of lines contains the explanation of the different behaviour of short and long waves.

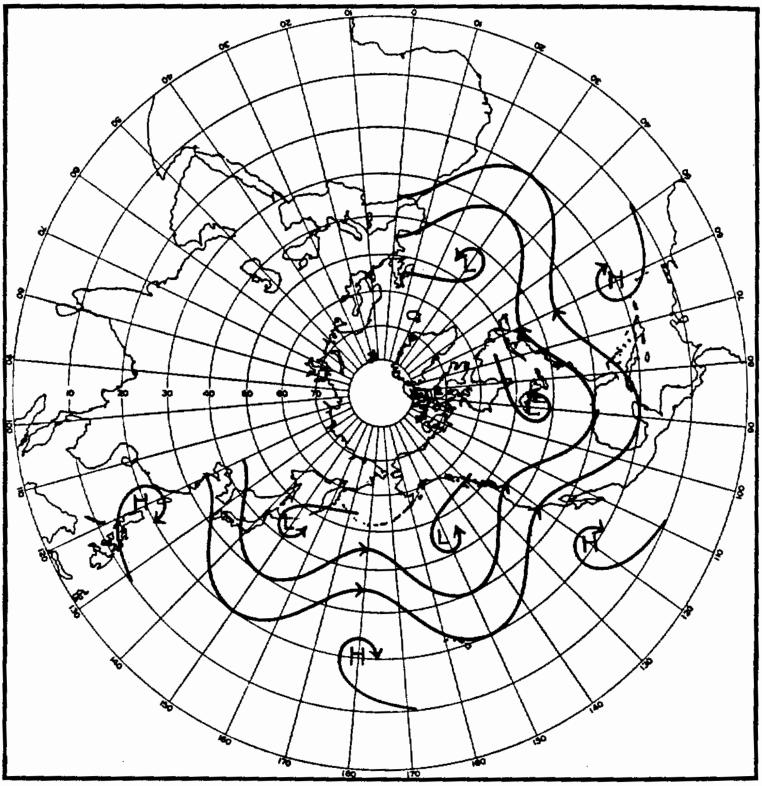

FIG. 4

Stationary horizontal flow patterns superimposed on barotropic atmosphere in steady zonal circulation. The undisturbed Easterlies to the North have a velocity of 8.9 mps., the undisturbed Easterlies to the South a velocity of 13.3 mps. The Westerlies in middle latitudes have a velocity of 15.5 mps. This last velocity gives a stationary wave length of sixty longitude degrees.

In order to achieve a certain amount of verisimilitude, suitably chosen flow patterns of the types computed in the present paper have been drawn on a chart for the Northern Hemisphere (Fig. 4). It has been assumed that steady west winds prevail in middle latitudes and the mean west wind velocity ($\bar{c}$) has been so chosen as to give a stationary wave length of 60° of longitude. The corresponding mean wind velocity is 15.5 m.p.s. in latitude 32.5°N. The two belts of easterly winds far to the north and to the south will, because of their dynamic instability, break up in eddies. The dimensions of the cyclonic eddies drawn to the north ($h = 10°$ of latitude) require an undisturbed current velocity of 8.9 m.p.s., and the anticyclonic eddies to the south ($h = 10°$ of latitude) require an undisturbed east wind velocity of 13.3 m.p.s. About the only thing that can be said is that these velocities are reasonable, and that the dimensions of the flow patterns thus computed correspond in a fairly satisfactory manner to the appearance of the atmosphere during periods of fairly weak circulation.

Further progress toward a satisfactory interpretation of the observed mean flow patterns in the atmosphere requires that depth changes and thermal effects resulting from the distribution of land and sea be taken into account.

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